Definition 0.1. Increasing returns to scale:

$$F(Ax, Ay) > AF(x, y), \tag{1}$$

where $A > 1$.

Definition 0.2. Decreasing marginal returns:

$$F(Ax, y) < AF(x, y), \tag{2}$$

where $A > 1$.

Decreasing marginal returns mathematically means:

$$\frac{\partial^2 F}{\partial x^2} < 0. \tag{3}$$

If we have decreasing marginal returns on two inputs:

$$\begin{aligned} F(Ax, y) < AF(x, y), \\ F(x, Ay) < AF(x, y), \end{aligned} \tag{4}$$

we can still have increasing returns to scale.

Example 0.3. Consider the Cobb-Douglas functional form:

$$F(K, L) = AK^{\alpha}L^{\beta}, \tag{5}$$

where $0 < \alpha < 1$ and $0 < \beta < 1$. Then, for $a > 1$,

$$F(aK, aL) = a^{\alpha+\beta}AK^{b}L^{c} = a^{\alpha+\beta}F(K, L), \tag{6}$$

it has IRS if $\alpha + \beta > 1$, CRS if $\alpha + \beta = 1$, and DRS if $\alpha + \beta < 1$.

But we have decreasing marginal returns in all the above 3 ways since, say for K,

$$\begin{aligned} \frac{\partial F}{\partial K} &= \alpha AK^{\alpha-1}L^{\beta} \\ \Rightarrow \frac{\partial^2 F}{\partial x^2} &= \alpha \times \underbrace{(\alpha - 1)}_{<0 \text{ since } 0<\alpha<1} \times AK^{\alpha-1}L^{\beta} < 0, \end{aligned} \tag{7}$$

which is not related to the sum of α and β.

Theorem 0.4. *If we have decreasing returns to scale, we must have two decreasing marginal returns.*

Proof. If we have decreasing returns to scale:

$$F(Ax, Ay) < AF(x, y), \tag{8}$$

where $A > 1$, then we must have

$$\begin{aligned} F(Ax, y) < \overbrace{F(Ax, Ay) < AF(x, y)}^{\text{by Equation 8}}, \\ \underbrace{F(x, Ay) < F(Ax, Ay)}_{\text{by monotonicity}} < AF(x, y), \end{aligned} \tag{9}$$

since the function F is increasing on both variables and x and y are both positive, then $Ax > x$ and $Ay > y$. □